On (k,l)-stable vector bundles over algebraic curves.

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Abstract

In this paper, we study the (k,l)-stable vector bundles over non-singular projective curve X of genus $g \geq 2$, its relation with stability and Segre invariants. For rank 2 and 3, we give an explicit description and relation of (k,l)-stability and Brill-Noether loci.

1 Introduction.

In [8, 9] Narasimhan and Ramanan introduced the notion of (k,l)-stability for vector bundles over a non-singular projective curve X of genus $g \geq 2$. A vector bundle E is (k,l)-stable if for all proper subbundle $F \subset E$ the differences of the slopes $\mu(E) - \mu(F)$ is greater than a rational number that involves k and l (see Definition 2.1). Mainly they use $(0,1)$, $(1,0)$ and $(1,1)$-stability to define an open set in the moduli space $M(n,L)$ of stable bundles over X with fix determinant L that allows them to define the Hecke cycles and Hecke curves. Also compute some cohomology groups of $M(n,L)$.

In this paper we study the (k,l)-stability for all $k,l \in \mathbb{Z}$. Denote by $A_{(k,l)}(n,d)$ the set of isomorphic classes of (k,l)-stable vector bundles of rank n and degree d over X. The non-emptiness conditions of $A_{(k,l)}(n,d)$ for any pair (k,l) of integers are given as follows: if $k(n-1)+l < (n-1)(g-1)$ and $k+(n-1)l < (n-1)(g-1)$ then $A_{(k,l)}(n,d) \neq \emptyset$ (Proposition 2.4). This bound could be improve for fix values of d and g (Theorem 2.7). Moreover, we obtain that whenever $A_{(k,l)}(n,d) \neq \emptyset$, there exist a stable vector bundle $E \in A_{(k,l)}(n,d)$.

If k,l satisfies $0 < k(n-1)+l$ and $0 < k+(n-1)l$, then every (k,l)-stable vector bundle is stable, i.e. $A_{(k,l)}(n,d) \subset M(n,d)$. In this case we compute the

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dimension and codimension of the complement of $A_{(k,l)}(n,d)$ in $M(n,d)$ (see, Theorem 2.12).
For others values of k, l is possible to have semistable vector bundles which are (k,l)-stables. For example, if $k(n-1)+l<0$ and $k+(n-1)l<0$, then every semistable vector bundle is (k,l)-stable (Proposition 2.9), in particular this holds for k, l negatives.
Let E and F be S-equivalent vector bundles and suppose that E is (k,l)-stable. Then it does not implies that F is (k,l)-stable (see Remark 2.13). In general, the (k,l)-stability splits the elements in the S-equivalence class of strict semistable vector bundles.
If (k,l) is such that $k(n-1)+l<0$ and $k+(n-1)l<0$, then there are unstable vector bundles (actually with automorphisms) that are (k,l)-stable. However, there exist decomposable unstable vector bundles which are not (k,l)-stable. If we consider indecomposable vector bundles only, then is possible to obtain conditions over k, l for which every indecomposable vector bundle is (k,l)-stable (Theorem 3.4).

Using the above results we make explicit computations for rank 2 and 3 cases. We give the necessary and sufficient conditions for non-emptiness of $A_{(k,l)}(n,d)$ (see, Theorem 3.2 and Table 1). Especially, in rank 2 case, we prove that every indecomposable vector bundle is (k,l)-stable if $k+l<2-2g$ (see Theorem 3.4), hence we obtain a complete classification of (k,l)-stable vector bundle of rank 2.
Moreover, for rank 3 we study the relation between semistability and (k,l) stability and we give the splitting of S-equivalence classes using (k,l)-stability (see, Theorem 4.1). Finally we apply of (k,l)-stability on Brill-Noether theory. We study the relation between $A_{(k,l)}(n,d)$ and $B(n,d,r)$ (see, Theorem 5.2)

This paper is as follows: Section 2 presents some basic properties and known results. We give necessary and sufficient conditions for non-emptiness of $A_{(k,l)}(n,d)$ and we compute of codimension for (k,l)-stable vector bundles of general rank. Section 3 establish the results for rank 2 case and Section 4 the results for rank 3 case. In Section 5 we relate the (k,l)-stability and the Brill-Noether loci.

2 (k,l)-stability.

From now on, X denotes a non-singular projective curve of genus $g \geq 2$ over $\mathbb{C}$. In this section we recall basic properties of (k,l)-stability, the proofs of some of the results can be found in [9].

For any integer k, the k-slope of a vector bundle E on X is the quotient

$$\mu_k(E) := \frac{deg\, E + k}{rk\, E}.$$

Definition 2.1. Let E be a vector bundle over X and $k,l \in \mathbb{Z}$. Then E is a (k,l)-stable vector bundle if for all proper subbundle $F \subset E$ we have $\mu_k(F) < \mu_{k-l}(E)$, i.e.

$$\frac{deg\, F + k}{rk\, F} < \frac{deg\, E + k - l}{rk\, E}.$$

If the inequality is not strict then E is (k,l)-semistable.

Let us denote by $A_{(k,l)}(n,d)$ the set of isomorphic classes of (k,l)-stable vector bundles of rank n and degree d over X.

The inequality in Definition 2.1, is equivalent to

$$(k(n-m)+ml)/nm < \mu(E) - \mu(F). \tag{2.1}$$

Remark 2.2. An easy computation shows the following statements:

1. If $(k,l) = (0,0)$, then $A_{(k,l)}(n,d) = M(n,d)$.
2. If $E \in A_{(k,l)}(n,d)$, then $E^* \in A_{(l,k)}(n,-d)$.
3. If $E \in A_{(k,l)}(n,d)$ and L a line bundle of degree d_L, then $E \otimes L \in A_{(k,l)}(n, d+nd_L)$.
4. If $k,l \geq 0$, then $A_{(k,l)}(n,d) \subseteq M(n,d)$.
5. If $k,l \leq 0$, then $A_{(k,l)}(n,d) \supseteq M(n,d)$.

It is known that the (k,l)-stability is an open property (see [9], Proposition 5.3). Hence by Remark 2.2, (4), if $k,l \geq 0$, then $A_{(k,l)}(n,d)$ is an open variety of the moduli space $M(n,d)$.

Another important property of (k,l)-stability is its behavior under elementary transformations. In this sense [9, Lemma 5.5] state that if $E \in A_{(k,l)}(n,d)$, $x \in X$ and $0 \to \underline{E'} \to \underline{E} \to \mathcal{O}_x \to 0$ is an exact sequence of sheaves with $\underline{E'}, \underline{E}$ locally free. Then $E' \in A_{(k,l-1)}(n, d-1)$. We reproduce the proof of this for the convenance of the reader.

Proof. Let $F' \subset E'$, and $F \subset E$ the generated bundle by the map $F \to E$. Then $F' \to F$ is of máximal rank and hence $deg\, F' \leq deg\, F$. Now $\mu_k(F') \leq \mu_k(F) < \mu_{k-l}(E) = \mu_{k-l+1}(E')$.

□

Remark 2.3. From definition of (k,l)-stability and using the inequality (2.1), we can observe that if E is a (k,l)-stable vector bundle, then E is $(k,l-1)$-stable and $(k-1,l)$-stable. Thus we have the following filtration

$$A_{(k,l)}(n,d) \subseteq A_{(k,l-1)}(n,d) \subseteq A_{(k,l-2)}(n,d) \subset \cdots$$

$$A_{(k,l)}(n,d) \subseteq A_{(k-1,l)}(n,d) \subseteq A_{(k-2,l)}(n,d) \subset \cdots$$

The conditions for non-emptiness of $A_{(0,1)}(n,d)$, $A_{(1,0)}(n,d)$, and $A_{(1,1)}(n,d)$ are given by Narasimhan and Ramanan in [9, Proposition 5.4], this is

1. Except when $g=2,\ n=2$ and d odd, $A_{(0,1)}(n,d) \neq \emptyset$.
2. $A_{(1,1)}(n,d) \neq \emptyset$ except in the following cases:

   (a) $g=3$, and d both even.
   (b) $g=2,\ d \equiv 0,\ \pm 1 (\mathrm{mod}\ n)$.
   (c) $g=2,\ n=4,\ d \equiv 2 (\mathrm{mod}\ 4)$.

We study the non-emptiness conditions for $A_{(k,l)}(n,d)$ with any value of k and l. Observe that if $A_{(k,l)}(n,d)$ is non empty, then $A_{(k,l-1)}(n,d)$ and $A_{(k-1,l)}(n,d)$ are non empty (Remark 2.3). By this reason we will prove the non-emptiness for k and l bigger enough. Following the idea of Narasimhan and Ramanan we obtain the following result, which implies [9, Proposition 5.4].

Proposition 2.4. *If $k,l \in \mathbb{Z}$ are such that*

$$k(n-1)+l < (n-1)(g-1) \tag{2.2}$$

and

$$k+l(n-1) < (n-1)(g-1) \tag{2.3}$$

then $A_{(k,l)}(n,d) \neq \emptyset$.

Proof. If (k,l) satisfies (2.2) and (2.3), we will prove that there exist stable vector bundles that are (k,l)-stable. Let $E \in M(n,d)$ be a not (k,l)-stable vector bundle on X of rank n degree d. Hence by definition there is a proper subbundle $F \subset E$ such that

$$\mu_{k-l}(E) \leq \mu_k(F), \tag{2.4}$$

and F determine the following extension $0 \to F \to E \to E/F \to 0$. If m and δ denote the rank and degree of F, then the number of such extensions is bounded

by $m^2(g-1)+1+(n-m)^2(g-1)+1+h^1\left((E/F)^*\otimes F\right)-1$. Now, using (2.2), (2.3) and (2.4) we obtain

$$m^2(g-1)+1+(n-m)^2(g-1)+1+h^1\left((E/F)^*\otimes F\right)-1 < n^2(g-1)+1.$$

This implies that the dimension of vector bundles which satisfies (2.4) is less than $\dim M(n,d)$. Therefore, the dimension of no (k,l)-stable vector bundles is less than $\dim M(n,d)$, this is for any $m=1,\dots,n-1$. As m take a finite values then does not cover the moduli space $M(n,d)$. An this proves the proposition.

□

Hence, we can rephrase [9, Proposition 5.3] as follows:

Corollary 2.5. *Under the hypotheses of Proposition 2.4, the very general vector bundle in $M(n,d)$ is (k,l)-stable.*

To obtain the pairs (k,l) such that $A_{(k,l)}(n,d)=\emptyset$, we will give a brief discussion about Segre invariants. The classical work of Segre invariant in rank 2 case is [6]. For a general treatment we refer the reader to [5]. A more complete theory in general rank may be obtained in [2, 10].

Let E be vector bundle on X of rank n, degree d and $m\in\mathbb{Z}$ such that $1\le m\le n-1$. Recall that the *m-Segre invariant* for a vector bundle E is denoted by $s_m(E)$ and defined as

$$s_m(E)=md-n\cdot deg\,F_{\max},$$

where $F_{\max}\subset E$ is a proper subbundle of rank m and maximal degree. Clearly, $s_m(E)\equiv md \mod n$.

Hirschowitz proved in [4] that,

$$s_m(E)\le m(n-m)(g-1)+(n-1). \tag{2.5}$$

Moreover, let X be a curve of genus g and let E be a vector bundle of rank n and degree d. There is an unique integer δ_m with $0\le\delta_m\le n-1$ and $m(n-m)(g-1)+\delta_m\equiv md \mod n$, such that

$$s_m(E)\le m(n-m)(g-1)+\delta_m. \tag{2.6}$$

The equality holds if E is general.

Denote by $M(n,d,m,s)$ the set of stable vector bundles of rank n and degree d such that the m-Segre invariant is s, that is

$$M(n,d,m,s):=\{E\in M(n,d)|s_m(E)=s\}. \tag{2.7}$$

From [10, Theorem 0.1] (see, [2, Theorem 4.2]) we have that if s is an integer such that, $0 < s \leq m(n-m)(g-1)$ and $s \equiv md \mod n$ and $g \geq 2$, then $M(n,d,m,s)$ is non-empty irreducible and

$$\dim M(n,d,m,s) = n^2(g-1) + 1 + s - m(n-m)(g-1).$$

Using above results we will prove the emptiness conditions of $A_{(k,l)}(n,d)$.

Proposition 2.6. *Let X be a non-singular projective algebraic curve of genus g. If $k, l \in \mathbb{Z}$ are such that*

$$k(n-1) + l \geq (n-1)g, \tag{2.8}$$

or

$$k + l(n-1) \geq (n-1)g, \tag{2.9}$$

then $A_{(k,l)}(n,d) = \emptyset$.

Proof. Let (k_0, l_0) be such that satisfies (2.8), we will prove that there does not exist any vector bundle which is (k_0, l_0)-stable.

Let E be a vector bundle of rank n and degree d and $L_0 \subset E$ a line subbundle of maximal degree, then by (2.5) and (2.8) respectively we obtain,

$$d - n \cdot deg\, L_0 = s_1(E) \leq (n-1)g \leq k_0(n-1) + l_0.$$

This implies $\mu_{k_0}(L_0) \geq \mu_{k_0 - l_0}(E)$, thus E is not (k_0, l_0)-stable vector bundle. Similarly if (k_0, l_0) satisfies (2.9) consider $F \subset E$ a subbundle of rank $n-1$ and maximal degree. This complete the proof.

□

The set of pairs (k, l) defined in Proposition 2.6 will be denoted by R_0, i.e.

$$R_0 := \left\{ (k,l) \in \mathbb{Z} \times \mathbb{Z} \;\middle|\; \begin{array}{l} k(n-1) + l \geq (n-1)g \quad \text{or} \\ k + l(n-1) \geq (n-1)g \end{array} \right\}. \tag{2.10}$$

Hence if $(k,l) \in R_0$ then $A_{(k,l)}(n,d) = \emptyset$.

However the bound given in Proposition 2.4 and Proposition 2.6, can be improved if we consider the degree, Segre invariants and the genus of the curve.

Theorem 2.7. *The set $A_{(k,l)}(n,d) \neq \emptyset$ if and only if the pair (k,l) is such that for all m, $1 \leq m \leq n-1$ the following inequality holds:*

$$k(n-m) + ml < m(n-m)(g-1) + \delta_m, \tag{2.11}$$

where δ_m is the unique integer (which depends of m) with $1 \leq \delta_m \leq n-1$ and $m(n-m)(g-1) + \delta_m \equiv md \mod n$.

Proof. ($\Rightarrow$) Let $E \in A_{(k,l)}(n,d) \neq \emptyset$, then combining equation (2.1), and (2.6) we have $k(n-m)+ml < s_m(E) \leq m(n-m)(g-1)+\delta_m$ for all m, this implies (2.11).
($\Leftarrow$) If (k,l) is such that for all m satisfies (2.11), then by (2.6) the generic vector bundle E is such that $m(n-m)(g-1)+\delta_m = s_m(E)$, for all m. Hence using equation (2.1), we conclude that $E \in A_{(k,l)}(n,d)$, and this complete the proof.

□

Remark 2.8. Observe that by inequality (2.1) the following statements holds:

1. If (k,l) is such that $k(n-m)+ml \geq 0$ for all m, $1 \leq m \leq n-1$. Then every (k,l)-stable vector bundle is stable.
2. If (k,l) be such that $0 \leq k(n-1)+l \leq (n-1)(g-1)$ and $0 \leq k+l(n-1) \leq (n-1)(g-1)$. Then $(k,l)-$stability implies stability.

The set of pairs that satisfies Remark 2.8, (2), will be denoted by R_1, that is

$$R_1 := \left\{ (k,l) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, \begin{array}{l} 0 \leq k(n-1)+l \leq (n-1)(g-1) \text{ and} \\ 0 \leq k+l(n-1) \leq (n-1)(g-1) \end{array} \right\}. \tag{2.12}$$

Hence, if $(k,l) \in R_1$, then $A_{(k,l)}(n,d) \neq \emptyset$.

Now we want to know when the stability implies (k,l)-stability. In this way it is easily seen that if for all m we have $k(n-m)+ml \leq 0$, then every stable vector bundle E is (k,l)-stable. For this, note that for all subbundle F of rank m we have, $\mu(E)-\mu(F) > 0 \geq (k(n-m)+ml)/nm$.

Proposition 2.9. *If (k,l) is such that $(n-1)k+l \leq 0$ and $k+(n-1)l \leq 0$. Then every stable vector bundle is (k,l)-stable.*

Proof. Let (k,l) be a pair of integer such that satisfies the conditions of proposition. We have divided the proof in three cases: First $k,l \leq 0$, second $k \leq 0$, $l > 0$ and third $k > 0$, $l \leq 0$.
If $k,l \leq 0$, it follows easily that every stable vector bundle is (k,l)-stable. Now, if $k \leq 0$ and $l > 0$, using the fact that $k+(n-1)l \leq 0$, we obtain

$$k(n-m)+ml \leq (l(1-n))(n-m)+ml = nl(m+1-n) \leq 0,$$

for all m. Hence, the assertion follows from Remark 2.8, (1). Similarly, if we suppose that $k > 0$, $l \leq 0$, then taking in count that $k(n-1)+ml \leq 0$ we have

$$k(n-m)+ml \leq k(n-m)+m(k(1-n)) = nk(1-m) \leq 0,$$

for all m. Thus for any pair (k,l) which satisfies the hypothesis, is such that $k(n-m)+ml \leq 0$, for all m. Now, combining this inequalities with Remark 2.1, (3), the proposition follows.

□

As above we define the following region.

$$R_2 := \left\{ (k,l) \in \mathbb{Z} \times \mathbb{Z} \,\middle|\, \begin{array}{l} (n-1)k + l \leq 0 \quad \text{and} \\ k + (n-1)l \leq 0 \end{array} \right\}. \tag{2.13}$$

The relation between (k,l)-stability and stability in the different regions described in the above propositions is rewrite as:

1. If $(k,l) \in R_0$, then $A_{(k,l)}(n,d) = \emptyset$.
2. If $(k,l) \in R_1$, then $A_{(k,l)}(n,d) \subset M(n,d)$.
3. If $(k,l) \in R_2$, then $M(n,d) \subset A_{(k,l)}(n,d)$.

We mentioned above that (k,l)-stability is an open property. Thus, if $A_{(k,l)}(n,d) \subset M(n,d)$ then $A_{(k,l)}(n,d)$ has dimension $n^2(g-1)+1$. The relation between $A_{(k,l)}(n,d)$ and $M(n,d)$ is as follows.

Proposition 2.10. *Over an algebraic curve X of genus $g \geq 2$, if $(k,l) \in R_1$ then*

$$A_{(k,l)}(n,d) = \bigcap_{m=1}^{n-1} \left(\bigcup_{s > k(n-m)+ml} M(n,d,m,s) \right).$$

Proof. It is easily seen that if $E \in A_{(k,l)}(n,d)$, then $s_m(E) > k(n-m)+ml$ for all $1 \leq m \leq n-1$, which implies the contention $(\subseteq)$.
Now suppose that

$$E \in \bigcap_{m=1}^{n-1} \left(\bigcup_{s > k(n-m)+ml} M(n,d,m,s) \right),$$

thus for all m, $s_m(E) > k(n-m)+ml$. This implies that $E \in A_{(k,l)}(n,d)$ and the proof is complete.

□

Now, if we denote by $A^c_{(k,l)}(n,d)$ the complement of $A_{(k,l)}(n,d)$ in $M(n,d)$ then we compute its codimension.

Proposition 2.11. *If $(k,l) \in R_1$, then*

$$codim A^c_{(k,l)}(n,d) \geq \min \left\{ \begin{array}{c} (n-1)(g-1) - k(n-1) - l, \\ (n-1)(g-1) - k - l(n-1) \end{array} \right\}.$$

Proof. Let $E \in A^c_{(k,l)}(n,d)$ and $F \subset E$ a subbundle of rank m and degree δ. As in proof of Proposition 2.4, the dimension of stable vector bundles which have a subbundle F of rank m and degree δ such that $\mu_{k-l}(E) > \mu_k(F)$ is $(n^2 - nm + m^2)(g-1) + 1 + dm - n\delta$. Moreover, this number is upper bounded by $(n^2 - nm + m^2)(g-1) + 1 + (n-m)k + ml$. Then

$$\dim M_X(n,d) - \dim(A^c_{(k,l)}(n,d)) \geq (nm - m^2)(g-1) - (n-m)k - ml.$$

Considering m as variable, we can see that the maximum of $(nm - m^2)(g-1) - (n-m)k + ml$ is obtained when $m = 1$ or $n-1$. Consequently, the codimension of $A^c_{(k,l)}(n,d)$ is lower bounded by

$$\min\{(n-1)(g-1) - k(n-1) - l,\ (n-1)(g-1) - k - l(n-1)\}.$$

This is the desired conclusion. $\square$

To compute explicitly the dimension and codimension of $A^c_{(k,l)}(n,d)$, we define the following variables:
Let

$$\widetilde{s_m} := \max\{s \mid s \leq k(n-m) + ml,\ s \equiv md \mod n\}. \tag{2.14}$$

$$s_\Delta := \min_m \{m(n-m)(g-1) - \widetilde{s_m}\}. \tag{2.15}$$

Theorem 2.12. *If $(k,l) \in R_1$, then*

1. $\dim A^c_{(k,l)}(n,d) = n^2(g-1) + 1 - s_\Delta$.
2. *codim*$A^c_{(k,l)}(n,d) = s_\Delta$.

Proof. (1) By Proposition 2.10, we have the following

$$\begin{aligned}
dim\left(A_{(k,l)}(n,d)\right)^c &= dim\left[\bigcap_{m=1}^{n-1}\left(\bigcup_{s>k(n-m)+ml} M(n,d,m,s)\right)\right]^c, \\
&= dim\left[\bigcup_{m=1}^{n-1}\left(\bigcup_{s>k(n-m)+ml} M(n,d,m,s)\right)^c\right], \\
&= dim\left[\bigcup_{m=1}^{n-1}\left(\bigcup_{s\leq k(n-m)+ml} M(n,d,m,s)\right)\right], \\
&= \max_m\left\{\max_s\left\{dim\left(M(n,d,m,s)\right)\right\}\right\}, \\
&= \max_m\left\{\max_s\left\{n^2(g-1)+1+s-m(n-m)(g-1)\right\}\right\}, \\
&= \max_m\left\{n^2(g-1)+1+\widetilde{s}_m-m(n-m)(g-1)\right\}, \\
&= n^2(g-1)+1-s_\Delta.
\end{aligned}$$

and this proves (1).
(2) As $A^c_{(k,l)}(n,d)$ is closed, the proof is straightforward from the difference

$$\begin{aligned}
dimM(n,d)-dimA^c_{(k,l)}(n,d) &= n^2(g-1)+1-\left(n^2(g-1)+1-s_\Delta\right), \\
&= s_\Delta.
\end{aligned}$$

□

2.1 Semistability and (k,l)-stability.

It is well known that, when degree and rank are coprime, semistability and stability coincide. Moreover, for semistable vector bundles there is an equivalence relation called S-equivalence. This equivalence relation is defined via the graduation of vector bundles which is obtained with the Jordan-Hölder filtration. Thus two vector bundles are S-equivalent if their graduations are isomorphic. However, is also possible that two vector bundles with different Jordan-Hölder filtration can be S-equivalents [11]. Therefore we want to use the (k,l)-stability in order to distinguish the strict semistable vector bundles, i.e. to determine the Jordan-Hölder filtration for each semistable vector bundle. We present this phenomena in the following example.

Example 2.13. *Consider E and E' two S-equivalent vector bundles of rank n and degree d. Suppose that $gr(E)=gr(E')=F_1\oplus F_2$ with $0\subset F_1\subset E'$ of rank n_1 and $0\subset F_2\subset E$ of rank n_2, with $n_1<n_2$.*

If $k \geq -l > 0$, then E' is not (k,l)-stable because $\mu_k(F_1) < \mu_{k-l}(E')$ implies $k < (n_1/n_2)(-l)$ which is a contradiction. Therefore if $E \in A_{(k,l)}(n,d)$, then (k,l)-stability split the class of S-equivalence of E.

More generally, the (k,l)-stability split the S-equivalence classes in the different types of Jordan-Hölder filtration. Each type of Jordan-Hölder filtration will correspond to a region in (k,l)-plane. Now, as a first step, we describe the following two regions named R_{3k} and R_{3l}.

$$R_{3k} := \left\{ (k,l) \,\middle|\, \begin{array}{c} 0 < k(n-1)+l < (n-1)(g-1) \quad \text{and} \\ k + l(n-1) < 0 \end{array} \right\}. \tag{2.16}$$

$$R_{3l} := \left\{ (k,l) \,\middle|\, \begin{array}{c} 0 < k+(n-1)l < (n-1)(g-1) \quad \text{and} \\ k(n-1) + l < 0 \end{array} \right\}. \tag{2.17}$$

Remark 2.14. The regions are defined considering the values (k,l). By definition, if $(k,l) \in R_{3k}$, then $k > 0, l < 0$ and if $(k,l) \in R_{3l}$ then $l > 0, k < 0$.

In both cases the first inequality in (2.16) (respectively (2.17)), is to consider the non-emptiness given by Proposition 2.4. Using both regions we define R_3 as the union, i.e.

$$R_3 := R_{3k} \cup R_{3l}. \tag{2.18}$$

Hence R_3 is the region that determine the relation between (k,l)-stability and the Jordan-Hölder filtration (see, Figure 1).

For $n = 3$, the regions R_{3k} and R_{3l} will split the graduation in Jordan-Hölder filtration for strict semistable vector bundle of rank 3 (see, Theorem 4.1). This is because the graduation of a strict semistable vector bundle is $L_1 \oplus L_2$ or $L \oplus F$. Now, if $gr(E) = L_1 \oplus L_2$, then the Jordan-Hölder filtration is $0 \subset L_i \subset F \subset E$ for $i = 1$ or 2 and E is not (k,l)-stable if $(k,l) \in R_3$. Moreover, if $gr(E) = L \oplus F$, then the Jordan-Hölder filtration is $0 \subset L \subset E$ or $0 \subset F \subset E$. The first one implies that $E \notin A_{(k,l)}(n,d)$ if $(k,l) \in R_{3k}$. The second one implies that $E \notin A_{(k,l)}(n,d)$ if $(k,l) \in R_{3l}$.

For $n \geq 4$, we need subdivide R_{3k} and R_{3l} in more regions in order to classify the different types. Such subdivision is given by the lines $k(n-m) + ml = 0$, with $1 \leq m \leq n-1$. In Section 4 we describe the rank 3 case and the ideas that we use there, can be easily generalized for $n \geq 4$.

3 Rank 2 case.

In this section we describe the above results for rank 2. By inequality (2.1) for rank 2 case of (k,l)-stability depends of the sum $k+l$ only. That is, a vector

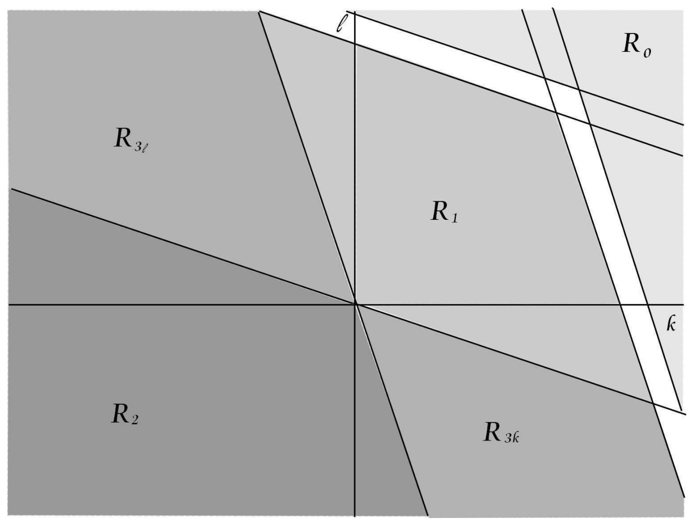

Figure 1: Regions defined by (k, l)-stability.

bundle E of rank 2 is (k, l)-stable if for any line subbundle $L \subset E$ satisfies

$$\mu(E) - \mu(L) > \frac{k+l}{2}.$$

Remark 3.1. In order to simplify notation we will write $A_t(2, d) := A_{(k,l)}(2, d)$, when $k + l = t$, and will be called t-stable instead (k, l)-stable. Using this notation (by Theorem 2.7), we have that:

1. If $t = 0$, then $A_t(2, d) = M(2, d)$
2. If $t > 0$ then $A_t(2, d) \subseteq M(2, d)$.
3. If $t = t'$, then $A_t(2, d) = A_{t'}(2, d)$.
4. If $t > t'$, then $A_t(2, d) \subseteq A_{t'}(2, d)$.

Moreover, in this case the Proposition 2.4, Proposition 2.6 and Theorem 2.7 are combined to obtain:

Theorem 3.2. *If X be a non-singular projective curve of genus g. Then we have the following statements.*

1. *For $g \not\equiv d \mod 2$, $A_t(2, d) \neq \emptyset$ if and only if $t < g - 1$.*
2. *For $g \equiv d \mod 2$, $A_t(2, d) \neq \emptyset$ if and only if $t \leq g - 1$.*

Proof. $(1, \Rightarrow)$ Let $E \in A_t(2,d)$ and let $L \subset E$ be a line subbundle of maximal degree, hence $t < d - 2\,deg\,L = s_1(E)$. Moreover, if $g \not\equiv d \mod 2$, it follows that $s_1(E) \neq g$. Thus, using (2.5) we have that $t < g-1$ as required.
$(1, \Leftarrow)$ This implication is a consequence of Proposition 2.4 taking $n = 2$.
$(2, \Rightarrow)$ The proof is similar to $(1, \Rightarrow)$ considering $g \equiv d \mod 2$.
$(2, \Leftarrow)$ By Proposition 2.4, if $t < g-1$ then $A_t(2,d) \neq \emptyset$. Hence it is enough to show the implication when $t = g-1$.
Let $E \in A_{g-2}(2,d)$ and $L \subset E$ be a line subbundle, then $d - 2deg\,L > g-2$. By hypothesis $g - 1 \not\equiv d \mod 2$, which implies $d - 2deg\,L \neq g-1$. Hence $d - 2deg\,L > g-1$ and therefore $E \in A_{g-1}(2,d)$ which proves that $A_{g-1}(2,d)$ is non-empty.

□

The proof of $(2, \Leftarrow)$ gives more, namely $A_{g-2}(2,d) = A_{g-1}(2,d)$ if $g \equiv d \mod 2$. The following result makes explicit this relation.

Proposition 3.3.

1. *If d is even, $r \in \mathbb{Z}$ and $2r \leq g-1$, then $A_{2r}(2,d) = A_{2r+1}(2,d)$.*
2. *If d is odd, $r \in \mathbb{Z}$ and $2r+1 \leq g-1$, then $A_{2r+1}(2,d) = A_{2r+2}(2,d)$.*

Proof. By (4) in Remark 3.1 we only need to prove the contentions $\subseteq$.
$(1, \subseteq)$ Let $E \in A_{2r}(2,d)$, then $d - 2deg\,L \geq 2r+1$ for any line subbundle $L \subset E$. But $d - 2deg\,L$ is even which implies $d - 2deg\,L > 2r+1$. Therefore $E \in A_{2r+1}(2,d)$.
Similar arguments apply to prove (2).

□

Clearly, if $t < 0$ then every semistable vector bundle of rank 2 is t-stable. However this does not apply to unstable case, because it is possible construct a unstable vector bundle which is not t-stable. For this choose any $t_0 < 0$ and consider two line bundles L_1 and L_2 such that $deg\,L_2 - deg\,L_1 \leq t_0$. Now define E as $E = L_1 \oplus L_2$. Hence E is unstable vector bundle such that is not t_0-stable as we desire.

Thus, if we consider only the indecomposable case, then is possible establish a lower bound such that every indecomposable vector bundle of rank 2 is t-stable.

Theorem 3.4. *If E is an indecomposable vector bundle of rank 2 and degree d then $E \in A_{1-2g}(2,d)$.*

Proof. If E is semistable then the proof is clear. Suppose that E is an indecomposable and unstable vector bundle such that $E \notin A_{1-2g}(2,d)$. Hence if L_0 is the line subbundle of maximal degree of E then $d - 2deg\,L_0 \leq 1 - 2g < 2 - 2g$, and we have the following extension

$$0 \to L_0 \to E \to L_1 \to 0.$$

Thus, $deg\, L_1 - deg\, L_0 = d - 2deg\, L_0 < 2-2g$, in consequence $d(L_1^* \otimes L_0) > 2g-2$ which implies $H^1(L_1^* \otimes L_0) = 0$, i.e. the extension splits. This contradicts the fact that E is indecomposable and proves the theorem.

□

Remark 3.5. The theorem asserts that if t is such that $t \leq 1-2g$, then all rank 2 degree d indecomposable vector bundle is t-stable. Moreover, if $r \in \mathbb{N}$, then we have the following contentions

1. If d is even then

$$\emptyset = A_g(2,d) \subset \cdots \subset A_{2r+1}(2,d) = A_{2r}(2,d) \subset \cdots$$
$$\cdots \subset A_1(2,d) = M(2,d) = A_0(2,d) \subset A_{-1}(2,d) \subset \cdots \subset A_{1-2g}(2,d).$$

2. If d is odd then

$$\emptyset = A_g(2,d) \subset \cdots \subset A_{2r+2}(2,d) = A_{2r+1}(2,d) \subset \cdots$$
$$\cdots = A_1(2,d) \subset A_0(2,d) = M(2,d) = A_{-1}(2,d) \subset \cdots \subset A_{1-2g}(2,d).$$

Now, we will describe in terms of t the different regions given in Section 2.1 for rank 2 case. By Theorem 3.2 we have two cases:

Case $g \equiv d \mod 2$.

1. $R_0 = \{t \mid t \geq g\}$ and $A_t(2,d) = \emptyset$ if and only if $t \in R_0$.
2. $R = \{t \mid t \leq g-1\}$ and $A_t(2,d) \neq \emptyset$ if and only if $t \in R$.
3. $R_1 = \{t \mid 0 \leq t \leq g-1\}$ and $A_t(2,d) \subset M(2,d)$ if $t \in R_1$.
4. $R_2 = \{t \mid t \leq 0\}$ and $A_t(2,d) \supset M(2,d)$ if $t \in R_2$.

Case $g \not\equiv d \mod 2$.

1. $R_0 = \{t \mid t \geq g-1\}$ and $A_t(2,d) = \emptyset$ if and only if $t \in R_0$
2. $R = \{t \mid t \leq g-2\}$ and $A_t(2,d) \neq \emptyset$ if and only if $t \in R$.
3. $R_1 = \{t \mid 0 \leq t \leq g-2\}$ and $A_t(2,d) \subset M(2,d)$ if $t \in R_1$.
4. $R_2 = \{t \mid t \leq 0\}$ and $A_t(2,d) \supset M(2,d)$ if $t \in R_2$.

Remark 3.6. In this case R_3 defined in (2.18), is empty when $n = 2$. The reason is that we have the line $k + l = 0$ only.

Moreover, it is possible compute explicitly the dimension and codimension of $A_t^c(2,d)$. In order to do this, combining (2.14) and (2.15) we have that

$$\widetilde{s_m} := \begin{cases} t, & \text{if } t \equiv d \, mod \, 2, \\ t-1, & \text{if } t \not\equiv d \, mod \, 2, \end{cases}$$

and

$$s_\Delta = \begin{cases} g-t-1, & \text{if } t \equiv d \, mod \, 2, \\ g-t-2, & \text{if } t \not\equiv d \, mod \, 2. \end{cases}$$

Now using Theorem 2.12 we obtain the following result.

Theorem 3.7. *If t be such that $t \in R_1$ then:*

$$\dim A_t^c(2,d) = \begin{cases} 3g+t-2, & \text{if } \; t \equiv d \mod 2, \\ 3g+t-3, & \text{if } \; t \not\equiv d \mod 2. \end{cases}$$

$$codim \, A_t^c(2,d) = \begin{cases} 3g+t-1, & \text{if } \; t \equiv d \mod 2, \\ 3g+t-2, & \text{if } \; t \not\equiv d \mod 2. \end{cases}$$

4 Rank 3 case.

Let E_1 and E_2 be two vector bundles of rank n and degree d, with n and d not coprime. If E_1 and E_2 are strictly semistable and S-equivalents, is false that the (k,l)-stability of E_1 implies the (k,l)-stability of E_2 (see, Example 2.13). Hence, in order to establish the behavior of (k,l)-stability in semistable vector bundles we study explicitly the relation of the (k,l)-stability in rank 3 and the Jordan-Hölder filtration.

Remember that in Section 2 we define the region R_3 as the union of R_{3l} and R_{3k} which are defined using the lines $k(n-1)+l=0$ and $k+l(n-1)=0$ (see, (2.16),(2.17) and (2.18)).

Theorem 4.1. *Let E be a vector bundle on X strictly semistable of rank 3. Then the Jordan-Hölder filtration of E is of one of this types:*

1. $0 \subset L \subset E$, *for some line bundle L if and only if there exist a pair $(k,l) \in R_{3l}$ such that E is (k,l)-stable.*

2. $0 \subset F \subset E$, *for some vector bundle F of rank 2 if and only if there exist a pair $(k,l) \in R_{3k}$ such that E is (k,l)-stable.*

3. $0 \subset L \subset F \subset E$, *for some line bundle L ans some vector bundle F of rank 2 if and only if E is not (k,l)-stable for all $(k,l) \in R_3$. Moreover, E is (k,l)-stable for $(k,l) \in R_2$*

Proof. For (1) the proof is based on the following observation. If the Jordan-Hölder filtration of E is of type $0 \subset L \subset E$, then every subbundle $G \subset E$ of rank 2 satisface $0 < \mu(E) - \mu(G)$. Moreover, $deg\, E$ is multiple of 3, for otherwise E will be stable.
$(1, \Rightarrow)$ Suppose the implication is false. Thus, E is not (k,l)-stable for all $(k,l) \in R_{3l}$. Taking an arbitrary point $(k_0, l_0) \in R_{3l}$, we have $0 < k_0 + 2l_0,\ 2k_0 + l_0 < 0$ by definition of R_{3l}. As E is not (k_0, l_0)-stable, there is a subbundle $G_0 \subset E$ of rank m such that

$$\mu(E) - \mu(G_0) \leq (k_0(3-m) + ml_0)/3m, \text{ with } m = 1 \text{ or } 2.$$

If $m = 1$ and $0 < \mu(E) - \mu(G_0) \leq ((2k_0 + l_0)/3)$ which is a contradiction to $2k_0 + l_0 < 0$.
It follows that $m = 2$, thus $0 < \mu(E) - \mu(G_0) \leq ((k_0 + 2l_0)/6)$, which implies $2\, deg\, E - 3\, deg\, G_0 \leq k_0 + 2l_0$. Now, as (k_0, l_0) is arbitrary we can choose it such that $k_0 + 2l_0 = 1$, but this imply that $deg\, E$ and 3 are coprime and this is a contradiction (remember that 3 divides to $deg\, E$). This conclude that E is (k,l)-stable for some $(k,l) \in R_{3l}$.
$(1 \Leftarrow)$ Note that is sufficient to prove that, if $G \subset E$ is a subbundle of rank 2 then $\mu(G) < \mu(E)$. However, as $E \in A_{(k_0,l_0)}(3,d)$ for some $(k_0,l_0) \in R_{3l}$ and $k_0 + 2l_0 > 0$ then,

$$\mu(E) - \mu(G) > (k_0 + 2l_0)/6,$$

establishes the desired conclusion. For (2) and (3) the proofs are similar.

□

To complete the description of $A_{(k,l)}(3,d)$ we need consider degree and genus module 3 (see, Theorem 2.7). To get a better idea we will fix in $g = 2$ and consider all the possible cases, that is: $d \equiv i \mod 3$, for $i = 0, 1, 2$ (compare with [9, Proposition 5.4]).

$\mathbf{d \equiv 0} \mod \mathbf{3}$. We will make the following computations. By (2.6) we know that if E is a rank 3, degree d vector bundle then $s_1(E) \leq 3$ and $s_2(E) \leq 3$. Hence $A_{(k,l)}(3,d) \neq \emptyset$ if and only if $2k + l < 3$ and $k + 2l < 3$. Hence $A_{(1,1)}(3,d) = \emptyset$, $A_{(0,1)}(3,d)$ and $A_{(1,0)}(3,d)$ are non-empty. Moreover, $A_{(0,1)}(3,0) = A_{(1,0)}(3,0) = M(3,0)$

$\mathbf{d \equiv 1} \mod \mathbf{3}$. As above using (2.6), we compute the bound for Segre invariants of E, $s_1(E) \leq 4$ and $s_2(E) \leq 2$. Hence $A_{(k,l)}(3,d) \neq \emptyset$ if and only if $2k + l < 4$ and $k + 2l < 2$. Hence $A_{(1,0)}(3,d) \neq \emptyset$ and $A_{(1,1)}(3,d)$, $A_{(0,1)}(3,d)$ are empty.

$\mathbf{d \equiv 2} \mod \mathbf{3}$. Similarly, $s_1(E) \leq 2$ and $s_2(E) \leq 4$. Hence $A_{(k,l)}(3,d) \neq \emptyset$ if and only if $2k + l < 2$ and $k + 2l < 4$. Therefore $A_{(0,1)}(3,d) \neq \emptyset$ and

$A_{(1,1)}(3,d)$, $A_{(1,0)}(3,d)$ are empty.

Remark 4.2. If we allow to vary the genus and following similar arguments we can obtain the necessary an sufficient conditions for $A_{(k,l)}(3,d) \neq \emptyset$. Table 1 consider the nine cases for rank 3 ($g, d \equiv 0, 1, 2 \mod 3$) and this complete the information about $A_{(k,l)}(3,d)$.
More general if $n \geq 4$, then it is necessary consider the n^2 possible cases of $d, g \equiv 0, 1 \ldots, n-1(\mod n)$.

	$g \equiv 0 \mod 3$	$g \equiv 1 \mod 3$	$g \equiv 2 \mod 3$
$d \equiv 0 \mod 3$	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g$, $k+2l < 2g$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-2$, $k+2l < 2g-2$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-1$, $k+2l < 2g-1$.
$d \equiv 1 \mod 3$	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-2$, $k+2l < 2g-1$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-1$, $k+2l < 2g$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g$, $k+2l < 2g-2$.
$d \equiv 2 \mod 3$	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-1$, $k+2l < 2g-2$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g$, $k+2l < 2g-1$.	$A_{(k,l)}(3,d) \neq \emptyset$ iff $2k+l < 2g-2$, $k+2l < 2g$.

Table 1: Non-emptiness for $A_{(k,l)}(3,d)$.

5 Application to Brill-Noether theory.

It is well known that there is a filtration of the moduli space $M(n,d)$ given by the Brill-Noether loci $B(n,d,r)$.

$$B(n,d,r) := \left\{ E \in M(n,d) | h^0(E) \geq r \right\}$$

We refer the reader to [1, 3, 7] for a general reference of the Brill-Noether theory.

Let μ and λ denote the quotients $\mu = d/n$ and $\lambda = r/n$. In this section we study the relation between $B(n,d,r)$ and $A_{(k,l)}(n,d)$. The interest in this relation is given by [2, Remark 4.5.].

Let $E \in B(n,d,r)$ be a (k,l)-stable vector bundle, i.e. $E \in B(n,d,r) \cap A_{(k,l)}(n,d)$. If we suppose that $\mathcal{O} \subset E$, then has sections and we have that

$$\frac{k(n-1)+l}{n} < \mu(E) - \mu(\mathcal{O}),$$

which implies $k(n-1)+l < d$. Then $E \notin A_{(k,l)}(n,d)$ if $l \geq d - k(n-1)$. Now, by Remark 2.3, $E \notin A_{(k,l)}(n,d)$ if $k \geq 0$, $l \geq d$. This prove the following

Proposition 5.1. *If $k \geq 0$, $l \geq d$ and $E \in B(n,d,r)$ is such that $\mathcal{O} \subset E$, then $E \notin A_{(k,l)}(n,d)$.*

The above result implies that if $\mathcal{O} \subset E$ and $E \in B(n,d,r)$, then we have $E \notin M(n,d,1,s_1)$ for $s_1 > d$. To have a better description of the relation between $B(n,d,r)$ and $A_{(k,l)}(n,d)$ we consider a different regions defined by the Brill-Noether theory.

(BGN). For $0 < \mu \leq 1$, then $B(n,d,r) \neq \emptyset$ if and only if

$$(\mu,\lambda) \in \{(\mu,\lambda) | 0 < \mu \leq 1,\ 1 \leq \mu + (1-\lambda)g,\ (\mu,\lambda) \neq (1,1)\}.$$

(M). If $1 < \mu < 2$, then $B(n,d,r) \neq \emptyset$ if and only if

$$(\mu,\lambda) \in \{(\mu,\lambda) | 1 \leq \mu + (1-\lambda)g\}.$$

(BMNO). If $d = nd' + d''$, $0 < d'' < 2n$, $0 \leq d'$ and $(d'', r) \neq (n,n)$, then $B(n,d,r) \neq \emptyset$. Moreover, if exists a line bundle L such that $h^0(L) \geq u$ with $1 \leq u \leq g$, then $B(n,d,ur) \neq \emptyset$.

Considering the information in BGN and M regions we relate the Brill-Noether loci with (k,l)-stability.

Theorem 5.2. *Let L a line bundle of degree d_L such that $h^0(L) = s$. Let $n \geq 2$, $E \in B(n,d,r)$ be a vector bundle and let $\mu = d/n$, $\lambda = r/n$. Then we have the following statements*

1. *If $(\mu,\lambda) \in BGN$, $k \geq 1$ and $l \geq 0$, then $E \notin A_{(k,l)}(n,d)$. Moreover, $E \otimes L \in B(n, d+nd_L, rs)$ and $E \otimes L \notin A_{(k,l)}(n, d+nd_L)$.*
2. *If $(\mu,\lambda) \in M$, $k \geq 2$, $l \geq 0$ and $d \neq 2n-1$, then $E \notin A_{(k,l)}(n,d)$. Moreover, $E \otimes L \in B(n, d+nd_L, rs)$ and $E \otimes L \notin A_{(1,0)}(n, d+nd_L)$.*

Proof. (1) Suppose that $E \in A_{(1,0)}(n,d)$, with $(\mu, \lambda) \in BGN$ hence $\mu(E) < 1$ and we have the following exact sequence

$$0 \to \mathcal{O} \to E \to Q \to 0. \tag{5.1}$$

Thus $(n-1)/n < \mu(E) - \mu(\mathcal{O}) = \mu(E) < 1$ which is impossible, consequently E is not $(1,0)$-stable. Moreover, if $k \geq 1$ and $l \geq 0$, then $A_{(k,l)}(n,d) \subset A_{(1,0)}(n,d)$ and therefore E is not (k,l)-stable (see, Remark 2.3). Using Remark 2.2, (3), it follows that $E \otimes L \notin A_{(k,l)}(n, d + nd_L)$. This prove the first statement.

(2) Suppose that $E \in A_{(2,0)}(n,d)$, then there is a line subbundle $L \subset E$ with sections such that $0 \leq deg(L) \leq 1$. Combining the $(2,0)$-stability of E with the hypothesis over μ we obtain,

$$\frac{2(n-1)}{n} < \mu(E) - \mu(L) = \mu(E) < 2.$$

However this implies $2n - 2 < d(E) < 2n$, which is a contradiction. In consequence, $E \notin A_{(2,0)}(n,d)$ and by Remark 2.3 we have $E \notin A_{(k,l)}(n,d)$ with $k \geq 2$ and $l \geq 0$. Using Remark 2.2, (3), it follows that $E \otimes L \notin A_{(k,l)}(n, d + nd_L)$. This prove the second statement.

$\square$

Now, we can rewrite the above result for rank 2 case. For this remember that that the (k,l)- stability depends of the sum $k + l$. Hence we using the notation of t-stability given in Section 3, it follows easily that if $t \geq d$, then $E \notin A_t(2,d)$. Moreover, when $s_1 > d$ we have $E \notin M(2, d, 1, s_1)$. Hence, from Theorem 5.2 we obtain the following result.

Corollary 5.3. *Let $E \in B(2,d,r)$ be a vector bundle. If $E \in BGN$, then $E \notin A_t(2,d)$ for all $t \geq 1$.*

Proposition 5.4. *Let $E \in B(2,d,1)$ such that $1 < \mu(E) < 2$. If E is 1-stable, then $E \in M(2,3,1,3)$.*

Proof. By hypothesis, $d = 3$. There is a line subbundle with sections $L \subset E$, then $0 \leq d(L) < 3/2$. By 1-stability we have $1/2 < \mu(E) - d(L) = (3/2) - d(L)$. Thus, $d(L) = 0$ and $s_1(E) = d$, which is the desire conclusion. $\square$

In the same sense, we have the same result for rank 3. This is, if $E \in B(3,d,r)$ determine a point in BGN, then $E \notin A_{(1,0)}(3,d)$. Moreover, if $E \in B(3,d,r)$ in M, then for $d = 4, 5$. Thus for $d = 4$, $E \notin A_{(2,0)}(3,4)$ but for $d = 5$ there are many possible results. Therefore we have the following existence for 1-stable vector bundles.

Let $E \in B(3,d,1)$ with $3 \leq d \leq 5$ and $E \in A_{(1,0)}(3,d)$. If $L \subset E$ is a maximal line subbundle, then $0 \leq d(L)$. Using $(1,0)$-stability of E, we have

that $1 \leq d(L)+1 < (d+1)/3 \leq 2$, which implies $d(L) = 0$. Therefore $s_1(E) = d$ and using notation of Segre invariants (see, (2.7)) we can see that

$$E \in M(3,3,1,3) \cup M(3,4,1,4) \cup M(3,5,1,5).$$

Now, as E is $(1,0)$-stable, then $s_2(E) \geq 2$. We thus get the following result.

Proposition 5.5. *Let $E \in B(3,d,1)$ be a vector bundle such that $1 \leq \mu(E) < 2$. If E is $(1,0)$-stable, when one of the cases holds:*

1. $E \in M(3,3,1,3) \cap M(3,3,2,s)$, *for some* $s \geq 2$,
2. $E \in M(3,4,1,4) \cap M(3,4,2,s)$, *for some* $s \geq 2$,
3. $E \in M(3,5,1,5) \cap M(3,5,2,s)$, *for some* $s \geq 2$.

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